All of Statistics - Chapter 1 Solutions

Apr 30, 2020

Acknowledgements: Thanks to Ben S. for correcting some mistakes.

1.

Let $i < j$. Since $B_i \subset A_i$ and $B_j \cap A_i = \emptyset$, it follows that B_i and B_j are disjoint.

Since $A_1 \subset A_2 \subset \cdots$, it follows that $A_n = \cup_{i=1}^n A_i$ for each n.

Suppose $\cup_{i=1}^n B_i = A_n$ for some n. By the previous claim, it follows that

$$\cup_{i=1}^{n+1} B_i = A_n \cup B_{n+1} = \left(\cup_{i=1}^n A_i\right) \cup \left(A_{n+1} \setminus \left(\cup_{i=1}^n A_i\right)\right) = \cup_{i=1}^{n+1} A_i.$$

Lastly, let $A_1 \supset A_2 \supset \cdots$ be monotone decreasing. Noting that $A_1^c \subset A_2^c \subset \cdots$ is monotone increasing,

$$\mathbb{P}(\cap_n A_n) = 1 - \mathbb{P}(\cup A_n^c) = 1 - \lim_n \mathbb{P}(A_n^c) = \lim_n 1 - \mathbb{P}(A_n^c) = \lim_n \mathbb{P}(A_n).$$

2.

Since $\mathbb{P}(\emptyset \cup \emptyset) = 2\mathbb{P}(\emptyset)$ by additivity, it follows that $\mathbb{P}(\emptyset) = 0$.

If A is contained in B, then

$$\mathbb{P}(B) = \mathbb{P}(A \cup B) = \mathbb{P}(A \cup (B \setminus A)) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) \geq \mathbb{P}(A)$$

As an immediate consequence of the previous two claims, it follows that $\mathbb{P}(A) \leq \mathbb{P}(\Omega) = 1$.

Since $\mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(A \cup A^c) = \mathbb{P}(\Omega) = 1$, it follows that $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$.

Lastly, we point out that by taking $A_2 = A_3 = \cdots = \emptyset$ in the countable additivity property (Axiom 3), we obtain finite additivity: $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2)$ for any disjoint sets A_1 and A_2.

3.

a)

Note that

$$B_n = \cup_{i=n}^\infty A_i \supset \cup_{i=n+1}^\infty A_i = B_{n+1}.$$

Similarly,

$$C_n = \cap_{i=n}^\infty A_i \subset \cap_{i=n+1}^\infty A_i = C_{n+1}.$$

b)

ω is in $\cap_n B_n \iff \omega$ is in B_n for each $n \iff$ for each n, we can find $i \geq n$ such that ω is in A_i.

Remark. A shorthand for $\cap_n B_n$ is $\limsup_n A_n$.

c)

ω is in $\cup_n C_n \iff \omega$ is in C_n for some $n \iff$ we can find n such that ω is in A_i for each $i \geq n$.

Remark. A shorthand for $\cup_n C_n$ is $\liminf_n A_n$.

4.

Note that

$$\begin{aligned} \omega \in (\cup_i A_i)^c &\iff \omega \notin \cup_i A_i \\ &\iff \omega \notin A_i \text{ for each } i \\ &\iff \omega \in A_i^c \text{ for each } i \\ &\iff \omega \in \cap_i A_i^c. \end{aligned}$$

Similarly,

$$\begin{aligned} \omega \in (\cap_i A_i)^c &\iff \omega \notin \cap_i A_i \\ &\iff \omega \notin A_i \text{ for some } i \\ &\iff \omega \in A_i^c \text{ for some } i \\ &\iff \omega \in \cup_i A_i^c. \end{aligned}$$

5.

The sample space for the repeated coin flip experiment is $\{H, T\}^{\mathbb{N}}$: the set of all functions from the natural numbers to $\{H, T\}$. Let X_n be one if the n-th toss is heads and zero otherwise. Then, the probability of stopping at the k-th toss is

$$\begin{aligned} \mathbb{P}(X_1 + \cdots + X_{k-1} = 1) \times \mathbb{P}(X_k = 1) = \binom{k-1}{1} p(1-p)^{k-2} \times p \\ = (k-1)\, p^2 (1-p)^{k-2}. \end{aligned}$$

The above simplifies to $(k-1)2^{-k}$ in the case of a fair coin.

6.

Let $\mathbb{P}$ be a probability measure on $\mathbb{N}$. By additivity, $1 = \mathbb{P}(\mathbb{N}) = \sum_n \mathbb{P}(\{n\})$. Suppose $\mathbb{P}$ is uniform. Then, $\mathbb{P}(\{n\}) = c$ for each n and hence $\mathbb{P}(\mathbb{N}) = c \cdot \infty$ (we interpret $0 \cdot \infty = 0$), a contradiction.

7.

Define B_n as in the hint. By our findings in Questions 1 and 2,

$$\mathbb{P}(\cup_n A_n) = \mathbb{P}(\cup_n B_n) = \sum_n \mathbb{P}(B_n) \leq \sum_n \mathbb{P}(A_n).$$

8.

Since

$$\mathbb{P}(\cup_i A_i^c) \leq \sum \mathbb{P}(A_i^c) = 0,$$

it follows that

$$\mathbb{P}(\cap_i A_i) = 1 - \mathbb{P}((\cap_i A_i)^c) = 1 - \mathbb{P}(\cup_i A_i^c) \geq 1 - 0 = 1.$$

9.

First, note that $\mathbb{P}(A \mid B) = \mathbb{P}(A \cap B)/\mathbb{P}(B) \geq 0$. In particular, $\mathbb{P}(\Omega \mid B) = 1$. Lastly, let $A_1, A_2, \ldots$ be disjoint. Then,

$$\mathbb{P}(\cup_n A_n \mid B) = \frac{\mathbb{P}(\cup_n (A_n \cap B))}{\mathbb{P}(B)} = \sum_n \frac{\mathbb{P}(A_n \cap B)}{\mathbb{P}(B)} = \sum_n \mathbb{P}(A_n \mid B).$$

10.

Without loss of generality, we can assume that the player picks door 1 and Monty reveals there is no prize behind door 2. Then, the player is left between choosing door $i = 1$ or $i = 3$. It follows that

$$p_i \equiv \mathbb{P}(\omega_1 = i \mid \omega_2 = 2) = \frac{\mathbb{P}(\omega_2 = 2 \mid \omega_1 = i)\mathbb{P}(\omega_1 = i)}{\mathbb{P}(\omega_2 = 2)} = \frac{\mathbb{P}(\omega_2 = 2 \mid \omega_1 = i)}{3\mathbb{P}(\omega_2 = 2)}.$$

In particular,

$$\mathbb{P}(\omega_2 = 2 \mid \omega_1 = i) = \begin{cases} 1/2 & \text{if } i = 1 \\ 1 & \text{if } i = 3. \end{cases}$$

Since the player should pick i to maximize p_i, the player should switch from door 1 to door 3.

11.

First, note that that

$$\mathbb{P}(A^c \cap B^c) = \mathbb{P}(A^c) - \mathbb{P}(B) + \mathbb{P}(A \cap B).$$

Using the independence of A and B,

$$
\begin{aligned}
\mathbb{P}(A^c \cap B^c) &= \mathbb{P}(A^c) - \mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(B) \\
&= \mathbb{P}(A^c) - (1 - \mathbb{P}(A))\, \mathbb{P}(B) \\
&= \mathbb{P}(A^c) - \mathbb{P}(A^c)\mathbb{P}(B) \\
&= \mathbb{P}(A^c)\,(1 - \mathbb{P}(B)) \\
&= \mathbb{P}(A^c)\mathbb{P}(B^c).
\end{aligned}
$$

12.

Let G_0 (respectively, G_1) be the event that the side of the seen (respectively, unseen) card is green. Since $\mathbb{P}(G_0) = 1/3 + 1/3 \cdot 1/2 = 1/2$, Then,

$$
\mathbb{P}(G_1 \mid G_0) = \frac{\mathbb{P}(G_0 \cap G_1)}{\mathbb{P}(G_0)} = \frac{1/3}{1/2} = 2/3.
$$

13.

a)

The sample space for this question is identical to that of Question 5.

b)

We stop at the third toss if and only if the first three flips are HHT or TTH. If p is the probability of heads, then the probability of this is $p^2(1-p) + (1-p)^2 p = p(1-p)$. In the case of a fair coin, this simplifies to $1/4$.

14.

Let A and B be events.

Suppose $\mathbb{P}(A) = 0$. Then, $\mathbb{P}(A \cap B) \leq \mathbb{P}(A) = 0$ and hence $\mathbb{P}(A \cap B) = 0 = \mathbb{P}(A)\mathbb{P}(B)$.

Suppose $\mathbb{P}(A) = 1$. Then, $\mathbb{P}(A^c) = 0$ and hence by our most recent findings, A^c and B^c are independent. By our findings in Question 11, it follows that A and B are independent.

Suppose now that A is independent of itself. Then, $\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A)$ and hence either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

15.

Let B_k be an indicator random variable that is one if and only if the k-th child has blue eyes. Let $B = B_1 + B_2 + B_3$. Let $p = 1/4$ be the probability of having blue eyes and $q = 1 - p$.

a)

Note that

$$\mathbb{P}(B \geq 2 \mid B \geq 1) = \frac{\mathbb{P}(B \geq 2)}{\mathbb{P}(B \geq 1)} = \frac{1 - \mathbb{P}(B \leq 1)}{1 - \mathbb{P}(B = 0)}.$$

Moreover, $\mathbb{P}(B = 0) = q^3$ and $\mathbb{P}(B = 1) = 3pq^2$. Therefore,

$$\mathbb{P}(B \geq 2 \mid B \geq 1) = \frac{1 - q^3 - 3pq^2}{1 - q^3} = \frac{10}{37}.$$

b)

Note that

$$\mathbb{P}(B \geq 2 \mid B_1 = 1) = \frac{\mathbb{P}(B_1 = 1, B_2 + B_3 \geq 1)}{\mathbb{P}(B_1 = 1)} = \mathbb{P}(B_2 + B_3 \geq 1)$$
$$= 1 - \mathbb{P}(B_2 + B_3 = 0) = 1 - q^2 = \frac{7}{16}.$$

16.

Let A and B be events with $\mathbb{P}(B) > 0$. $\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\mathbb{P}(B)$ follows by multiplying by $\mathbb{P}(B)$ on both sides of the definition of conditional probability. Moreover, if A and B are independent,

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A).$$

17.

Assuming $\mathbb{P}(BC)$ and $\mathbb{P}(C)$ are positive, the result follows from combining

$$\mathbb{P}(ABC) = \frac{\mathbb{P}(ABC)}{\mathbb{P}(BC)}\mathbb{P}(BC) = \mathbb{P}(A \mid BC)\mathbb{P}(BC)$$

and

$$\mathbb{P}(BC) = \frac{\mathbb{P}(BC)}{\mathbb{P}(C)}\mathbb{P}(C) = \mathbb{P}(B \mid C)\mathbb{P}(C).$$

18.

If $A_1, \ldots, A_k$ are a partition of the sample space, then $1 = \mathbb{P}(\cup_i A_i) = \sum_i \mathbb{P}(A_i)$. Moreover, for any event B,

$$\mathbb{P}(B) = \mathbb{P}((\cup_i A_i) \cap B) = \mathbb{P}(\cup_i (A_i \cap B)) = \sum_i \mathbb{P}(A_i \cap B).$$

If $\mathbb{P}(B) > 0$, then we can divide both sides by $\mathbb{P}(B)$ to get $1 = \sum_i \mathbb{P}(A_i \mid B)$. Combining this with a previous equality, we get $\sum_i \mathbb{P}(A_i) = \sum_i \mathbb{P}(A_i \mid B)$. Suppose now $\mathbb{P}(A_1 \mid B) < \mathbb{P}(A_1)$. Then,

$$\sum_{i \neq 1} \mathbb{P}(A_i) < \sum_{i \neq 1} \mathbb{P}(A_i \mid B).$$

It follows that $\mathbb{P}(A_i \mid B) > \mathbb{P}(A_i)$ for at least one i.

19.

We use M, W, and L to denote the event that the user uses Mac, Windows, and Linux, respectively. We use V to denote the event that the user has the virus.

$$\begin{aligned}\mathbb{P}(W \mid V) = \frac{\mathbb{P}(V \mid W)\mathbb{P}(W)}{\mathbb{P}(V)} &= \frac{\mathbb{P}(V \mid W)\mathbb{P}(W)}{\sum_{X \in \{M,W,L\}} \mathbb{P}(V \mid X)\mathbb{P}(X)} \\ &= \frac{82 \times 50}{65 \times 30 + 82 \times 50 + 50 \times 20} = \frac{82}{141} \approx 0.58.\end{aligned}$$

20.

a)

$$\mathbb{P}(C_i \mid H) = \frac{p_i \mathbb{P}(C_i)}{\mathbb{P}(H)} = \frac{p_i \mathbb{P}(C_i)}{\sum_j p_j \mathbb{P}(C_j)} = \frac{p_i}{\sum_j p_j}.$$

b)

$$\mathbb{P}(H_2 \mid H_1) = \frac{\mathbb{P}(H_1 \cap H_2)}{\mathbb{P}(H_1)} = \frac{\sum_i p_i^2 \mathbb{P}(C_i)}{\sum_i p_i \mathbb{P}(C_i)} = \frac{\sum_i p_i^2}{\sum_i p_i}.$$

c)

$$\mathbb{P}(C_i \mid B_4) = \frac{\mathbb{P}(C_i \cap B_4)}{\mathbb{P}(B_4)} = \frac{\mathbb{P}(B_4 \mid C_i)\mathbb{P}(C_i)}{\sum_j \mathbb{P}(B_4 \mid C_j)\mathbb{P}(C_j)} = \frac{(1 - p_i)^3 p_i}{\sum_j \left(1 - p_j\right)^3 p_j}.$$

21.

TODO (Computer Experiment)

22.

TODO (Computer Experiment)

23.

TODO (Computer Experiment)