Weakly chained matrices, policy iteration, and impulse control
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Agenda

Motivation: combined stochastic and impulse control

Some important matrix families

Policy iteration

The nonexpansive problem
Motivation: combined stochastic and impulse control
Controlled stochastic differential equation

- $\{\tau_j\}$ stopping times with $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \ldots \leq T$.
- $\mathcal{W}$ denotes a standard Brownian motion.
- $X$ solves a $d$-dimensional stochastic differential equation (SDE):
  \[
  dX_t = \mu(t, X_t, w_t) \, dt + \sigma(t, X_t, w_t) \, d\mathcal{W}_t \text{ on } \tau_j < t < \tau_{j+1}.
  \]
- $w$ is a càdlàg adapted stochastic control taking values in some compact set $W$.
- $\zeta_j$ is a $\tau_j$-adapted impulse taking values in the set $Z(\tau_j, X_{\tau_j^-})$.
  \[
  X_{\tau_j} = \Gamma(\tau_j, X_{\tau_j^-}, \zeta_j) \text{ for } j > 0.
  \]
- Combined stochastic and impulse control:
  \[
  \alpha = (w, \tau_1, \tau_2, \ldots, \zeta_1, \zeta_2, \ldots)
  \]
Value function

The value under a particular control $\alpha$ is

$$J^{(\alpha)} (t, x) = \mathbb{E}^{(t,x)} \left[ \int_t^T f (s, X_s^\alpha, w_s) \, ds \right.$$ 

$$+ g (X_T^\alpha) + \sum_{\tau_j \leq T} K \left( \tau_j, X_{\tau_j}^\alpha, \zeta_j \right).$$

The value function is the “best” value:

$$u (t, x) = \sup_{\alpha} J^{(\alpha)} (t, x).$$
Using a dynamic programming argument (see [Seydel, 2009]) and a comparison principle, $u$ can be shown\(^1\) to be the (unique) viscosity solution of

\[
\min \left( - \sup_{w \in \mathcal{W}} \left\{ \frac{\partial u}{\partial t} + \mathcal{L}^w u + f^w \right\}, u - \mathcal{M}u \right) = 0 \text{ on } [0, T) \times \mathbb{R}^d
\]

\[
\min \{ u - g, u - \mathcal{M}u \} = 0 \text{ on } \{ T \} \times \mathbb{R}^d
\]

where $\mathcal{L}^w$ is the infinitesimal generator of the SDE and

\[
\mathcal{M}u(t, x) = \sup_{\zeta \in Z(t,x)} \{ u(t, \Gamma(t, x, \zeta)) + K(t, x, \zeta) \}
\]

\(^1\)...subject to some technical conditions : -)
Partial order on $\mathbb{R}^M$ and $\mathbb{R}^{M \times M}$

**Definition**

Let $\leq$ denote the **element-wise partial order**. Denote by $\sup$ the supremum induced by this order.

**Example**

\[
\begin{pmatrix} 0 \\ 0 \end{pmatrix} < \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ but } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \nRightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

**Example**

\[
\sup_{\pi \in \mathbb{R}^2} \begin{pmatrix} -|\pi_1| \\ - (\pi_2)^2 + \pi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/4 \end{pmatrix}.
\]
A numerical scheme for the dynamic programming equation takes the form

\[
\text{find } v \in \mathbb{R}^M \text{ such that } \sup_{\pi \in \Pi} \{-A(\pi)v + b(\pi)\} = 0.
\]

(Bellman problem)

- Solved on a spatial grid with \( M \) nodes.
- \( v \) is the solution at each node.
- \( \pi = (\pi_1, \ldots, \pi_M) \in \Pi \) is a node-wise control.
- \( A(\pi) \in \mathbb{R}^{M \times M} \) (resp. \( b(\pi) \in \mathbb{R}^M \)) with \( i \)-th row depending only on \( \pi_i \).
Theorem

The problem

find $v \in \mathbb{R}^M$ such that $\sup_{\pi \in \Pi} \{-A(\pi)v + b(\pi)\} = 0$

(Bellman problem)

admits a unique solution whenever

- $A$ and $b$ are bounded and
- $I - A(\pi)$ is a contraction with Lipschitz constant bounded uniformly with respect to $\pi$.

Proof hint.

Write $v$ as the fixed point of some equation and apply the Banach fixed point theorem.
Motivation: combined stochastic and impulse control

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### Classical results (detailed proof)

**Proof.**

Adding $v$ to both sides of the equation yields

$$
\sup_{\pi \in \Pi} \left\{ (I - A(\pi)) v + b(\pi) \right\} = v.
$$

This suggests the fixed point iteration

$$
v^{k+1} = f(v^k) \text{ where } f(v) := \sup_{\pi \in \Pi} \left\{ (I - A(\pi)) v^k + b(\pi) \right\}.
$$

Moreover,

$$
|f(v) - f(v')| \leq \sup_{\pi \in \Pi} |(I - A(\pi))(v - v')| \leq L |v - v'|
$$

where $0 \leq L < 1$ is the Lipschitz constant of the previous slide.

Now, apply the Banach fixed point theorem.
The impulse term in
\[
\min \left( - \sup_{w \in \mathcal{W}} \left\{ \frac{\partial u}{\partial t} + \mathcal{L}_w u + f_w \right\}, u - \mathcal{M} u \right) = 0
\]
causes \( I - A(\pi) \) in the Bellman problem to be nonexpansive (and not necessarily contractive).
Some important matrix families
Diagonal dominance

Definition
Row $i$ of a complex matrix $A = (a_{ij})$ is strictly diagonally dominant (SDD) if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|.$$ 

We say $A$ is SDD if all of its rows are SDD.

Weakly diagonally dominant (WDD) is defined with $\geq$.

Example
The following matrix is SDD if $\epsilon > 0$ and WDD if $\epsilon = 0$:

$$\begin{pmatrix} 2 & 2 + \epsilon & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$
Matrix graph

Definition

The (directed) graph of an $M \times M$ complex matrix $A = (a_{ij})$ is given by $\text{graph}(A) = (V, E)$ where

$$V = \{1, \ldots, M\} \text{ and } E = \{(i, j): a_{ij} \neq 0\}.$$ 

Example

$$\begin{pmatrix}
2 & 2 + \epsilon & -1 \\
-1 & 2 & -1 \\
-1 & 2
\end{pmatrix}$$
Weakly chained diagonally dominant matrix

Definition
A complex square matrix \( A \) is said to be **weakly chained diagonally dominant (WCDD)** if

- \( A \) is WDD;
- each vertex in graph(\( A \)) has a path to an SDD vertex.

Example

\[
\begin{pmatrix}
2 & 2 & -1 \\
-1 & 2 & -1 \\
-1 & 2 & 1
\end{pmatrix}
\]
Another WCDD example

Example

\[
\begin{pmatrix}
1 & -1 \\
1 & -1 \\
\vdots & \vdots \\
1 & -1 \\
1 & 1
\end{pmatrix}
\]

1 → 2 → ··· → M
Lemma

A WCDD matrix is nonsingular.

Proof sketch (see [Azimzadeh and Forsyth, 2016] for a detailed proof).

- Suppose that there exists a WCDD matrix with eigenvalue \( \lambda = 0 \) and associated eigenvector \( v \neq 0 \).
- Pick \( i_0 \) such that \( |v_{i_0}| = \max_i |v_i| \).
- By the definition of WCDD, there exists a path \( i_0 \to i_1 \to \cdots \to i_k \) where \( i_k \) is an SDD vertex.
- Apply the Gershgorin circle theorem to establish that each vertex along this path is WDD; a contradiction.
Monotone matrix

Definition
A real square matrix $A$ is monotone (in the sense of Collatz) if for all real vectors $v$, $Av \geq 0$ implies $v \geq 0$.

To see the motivation behind this definition...

1. Let $F: \mathbb{R}^M \rightarrow \mathbb{R}^M$ be a function satisfying
   \[ x < y \implies F(x) < F(y). \]

2. The contrapositive of the above proposition is
   \[ F(x) \geq F(y) \implies x \geq y. \]

3. If in addition $F$ is linear,
   \[ F(x - y) \geq 0 \implies x - y \geq 0. \]
M-matrix

Definition

An **M-matrix** is a monotone matrix with nonpositive off-diagonals.

Theorem ([Azimzadeh and Forsyth, 2016])

*The following are equivalent:*

- A is a WDD M-matrix;
- A is a WCDD matrix with positive diagonals and nonpositive off-diagonals.

An M-matrix need not be WCDD:

\[
\begin{pmatrix}
1 & -2 \\
0 & 1
\end{pmatrix}
\]
Policy iteration
Policy iteration

Recall the Bellman problem:

\[
\text{find } v \in \mathbb{R}^M \text{ such that } \sup_{\pi \in II} \{-A(\pi)v + b(\pi)\} = 0
\]

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POLICY-ITERATION(\(v^0\))

1. for \(\ell = 1, 2, \ldots\)
2. Pick \(\pi^\ell\) such that
   \[
   -A(\pi^\ell)v^{\ell-1} + b(\pi^\ell) = \sup_{\pi \in II} \{-A(\pi)v^{\ell-1} + b(\pi)\}
   \]
3. Solve \(A(\pi^\ell)v^\ell = b(\pi^\ell)\) for \(v^\ell\)

Idea: pick a quasi-optimal policy, improve the value, and repeat until convergence.
Convergence of policy iteration

Theorem
Suppose
1. $\pi \mapsto A(\pi)^{-1}$, $A$, and $b$ are bounded,
2. for all $x \in \mathbb{R}^M$, the supremum of $\{-A(\pi)x + b(\pi)\}_{\pi \in \Pi}$ is attained at an element $\pi_x \in \Pi$, and
3. $A(\pi)$ is a monotone matrix for all $\pi \in \Pi$.

$(v^\ell)_{\ell \geq 1}$ defined by POLICY-ITERATION converges from below to the unique solution $v$ of the Bellman problem. Moreover, if $\Pi$ is finite, convergence occurs in at most $|\Pi|$ iterations.

In fact, we can even remove the second requirement by allowing the algorithm to pick the policy $\pi^\ell$ with some tolerable error $\epsilon^\ell$ such that $\epsilon^\ell \downarrow 0$ quickly enough (see appendix in [Azimzadeh and Forsyth, 2016]). In this case, convergence is only “nearly” from below.
The nonexpansive problem
SDD/WDD splitting

Assumption

Each policy $\pi$ is of the form

$$\pi = (\pi_1, \ldots, \pi_M) \text{ and } \pi_i = (\phi_i, \psi_i)$$

where $\phi = (\phi_1, \ldots, \phi_M)$ is arbitrary and $\psi_i \in \{0, 1\}$. $A(\pi)$ has nonpositive off-diagonals, and can be split as follows:

$$A(\pi) = (I - \text{diag}(\psi)) (I - L(\phi)) + \text{diag}(\psi) (I - B(\phi)).$$

The SDD and WDD parts correspond to the contractive and nonexpansive parts of $A(\pi)$.

The nonpositive off-diagonals occur naturally in applications (Markov chains and monotone discretizations of PDEs).
Convergence of policy iteration in the splitting problem

Theorem ([Azimzadeh and Forsyth, 2016])

Suppose

- $A$ and $b$ are bounded and
- for each $\pi = (\phi, \psi) \in \Pi$ and vertex $i$ with $\psi_i = 1$, there exists a path in the graph of $N(\phi)$ from $i$ to some vertex $j(i)$ with $\psi_{j(i)} = 0$.

$(v^\ell)_{\ell \geq 1}$ defined by Policy-Iteration converges from below to the unique solution $v$ of the Bellman problem.

Proof.
The above establishes that $A(\pi)$ is WCDD. Moreover,

$$\text{WCDD} \implies \text{M-matrix} \implies \text{monotone}.$$ 

Now we can apply a previous result to yield convergence.
Convergence under weaker conditions

The conditions of the previous theorem are too strong to cover all relevant cases.

**Insight:** policy iteration fails whenever $A(\pi^\ell)$ is singular.

We use the following idea:

1. establish uniqueness for the Bellman problem;
2. pick a subset $\Pi' \subset \Pi$ on which $A(\pi)$ is nonsingular and

$$\sup_{\pi \in \Pi'} \{-A(\pi)v + b(\pi)\} = 0 \implies \sup_{\pi \in \Pi} \{-A(\pi)v + b(\pi)\} = 0.$$

3. perform policy iteration on the modified problem.
Uniqueness under weaker conditions

Let $Bv = \sup_{\pi \in \Pi} \{B(\phi)v + b(\psi)\}$.

**Theorem ([Azimzadeh and Forsyth, 2016])**

Suppose

- $A$ and $b$ are bounded and
- for each solution $v$ of the Bellman problem and each vertex $i$, there exist integers $m(i)$ and $n(i)$ such that $0 \leq n(i) < m(i)$ and $[B^{m(i)}v]_i < [B^{n(i)}v]_i$.

A solution of the Bellman problem is **unique**.

**Proof sketch.**

1. Let $v$ be a solution with $-A(\pi^\ell)v + b(\pi^\ell) \nearrow 0$ as $\ell \nearrow \infty$.
2. Establish that $A(\pi^\ell)$ is WCDD (and hence monotone) for $\ell$ large enough.
3. Establish uniqueness using monotonicity.
Thank you for your attention :-}
Bibliography

Weakly chained matrices, policy iteration, and impulse control.

Existence and uniqueness of viscosity solutions for qvi associated with impulse control of jump-diffusions.