# Numerical combined stochastic and impulse control

## The 2015 AMMCS-CAIMS Congress

June 10, 2015

Parsiad Azimzadeh
`pazimzad@uwaterloo.ca`

David Cheriton School of Computer Science
University of Waterloo

# Agenda

Introduction

An example

Numerical schemes

Results

QuantPDE: open-source C++ library for pricing derivatives

# Introduction

## Stochastic process

### What is combined stochastic and impulse control?

- $\{\tau_j\}$ stopping times with $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \ldots \leq T$
- Standard Brownian motion $\mathfrak{W}$
- *d*-dimensional stochastic differential equation (SDE):

$$dX_t = \mu\left(t, X_t, q_t\right) dt + \sigma\left(t, X_t, q_t\right) d\mathfrak{W}_t \text{ on } \tau_j < t < \tau_{j+1}$$

- Impulses:

$$X_{\tau_j} = \Gamma\left(\tau_j, X_{\tau_j -}, \zeta_j\right) \text{ for } j > 0$$

- Control:

$$\alpha = \left(q, \tau_1, \tau_2, \ldots, \zeta_1, \zeta_2, \ldots\right)$$

## What control are we interested in?

- $\tau_S$ is the exit time of $X^\alpha$ from $S \subset \mathbb{R}^d$ (open) and $\tau = \min\{\tau_S, T\}$
- Value under $\alpha$:

$$J^{(\alpha)}(t,x) = \mathbb{E}^{(t,x)}\left[\int_t^\tau e^{-\rho s} f(s, X_s^\alpha, q_s)\,ds + e^{-\rho\tau} g(\tau, X_\tau^\alpha) 1_{\tau<\infty} + \sum_{\tau_j \leq \tau} e^{-\rho\tau_j} K\left(\tau_j, X_{\tau_j -}^\alpha, \zeta_j\right)\right]$$

- *Combined stochastic and impulse control* problem:

$$u(t,x) = e^{\rho t} \sup_\alpha J^{(\alpha)}(t,x)$$

## How do we solve these problems?

- Solution satisfies a *Hamilton-Jacobi-Bellman quasivariational inequality* (HJBQVI) in viscosity sense.
- Solutions (to HJBQVI) not necessarily unique
- Usually need to impose growth condition
- Added point of complexity: numerical methods may not converge to a viscosity solution

# Introduction

## HJBQVI (parabolic)

Find a viscosity solution $u$ of the HJBQVI

$$\min\left(-\sup_{q\in Q}\left\{\frac{\partial u}{\partial t}+Ł^{q}u-\rho u+f^{q}\right\},u-\mathcal{M}u\right)=0\text{ on }[0,T)\times S$$

$$\min\left\{u-g,u-\mathcal{M}u\right\}=0\text{ on }[0,T)\times\left(\mathbb{R}^{d}\setminus S\right)$$

$$u-g=0\text{ on }\{T\}\times\overline{S}$$

where $Ł^{q}$ is the infinitesimal generator of the SDE and

$$\mathcal{M}u\left(t,x\right)=\sup_{\zeta\in Z(t,x)}\left\{u\left(t,\Gamma\left(t,x,\zeta\right)\right)+K\left(t,x,\zeta\right)\right\}.$$

# An example

## Optimal control of the exchange rate

### Stochastic process

- Government influences foreign exchange (FEX) rate by:
  - choosing domestic interest rate (stochastic control);
  - intervening in FEX market by buying ($\zeta > 0$) or selling ($\zeta < 0$) foreign currency (impulse control).
- $X_t$ is the log of the FEX rate:

$$
\begin{aligned}
dX_t &= -aq_t dt + \sigma d\mathfrak{W}_t && \text{(stochastic control)} \\
X_{\tau_j} &= X_{\tau_j -} + \zeta_j && \text{(impulse control).}
\end{aligned}
$$

# An example

## Optimal control of the exchange rate

### Optimal cost

- $\lambda, c > 0$ parameterize cost of intervention in FEX market
- Penalize when (domestic) currency is weak ($X_t > 0$)
- Penalize for high interest rate
- Given by

$$u(t,x) = e^{\rho t} \sup_{\alpha} \mathbb{E}^{(t,x)} \Bigg[ - \int_t^T e^{-\rho s} \left( \max\{X_s, 0\}^2 + b q_s^2 \right) ds - \sum_{\tau_j \leq T} e^{-\rho \tau_j} \left( \lambda |\zeta| + c \right) \Bigg]$$

# An example

## Optimal control of the exchange rate

### HJBQVI

- Initial condition:

$$u(T, x) = 0 \text{ on } \mathbb{R}$$

  (regardless of the state $x$, when the game is over ($t = T$), no costs can be incurred)
- HJBQVI:

$$0 = \min\left( -\sup_{q \in Q} \left\{ \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} - aq\frac{\partial u}{\partial x} - \rho u - x^2 - bq^2 \right\}, \right.$$
$$\left. u - \sup_{\zeta \in \mathbb{R}} \left\{ u\left(t, x + \zeta\right) - \lambda \left|\zeta\right| - c \right\} \right) \text{ on } [0, T) \times \mathbb{R}$$

- If $c = 0$, $u = 0$ (everywhere) is a solution! (check)
- Not a solution of the original problem!

# Numerical schemes

## Numerical PDEs

- This talk focuses on numerical PDE methods
- Another method that is worth mentioning is constrained backward SDEs with Monte Carlo simulations: Kharroubi, Idris, et al. "Backward SDEs with constrained jumps and quasi-variational inequalities." *The Annals of Probability* 38.2 (2010): 794-840.

## Fully implicit

- Let Ł$_h$ denote a consistent discretization of Ł
- Standard approach: penalize with $\epsilon > 0$ if $u - \mathcal{M}u < 0$
- Fully implicit scheme:

$$u_i^n = \left( u_i^{n-1} + \sup_{q \in Q_h} \left\{ (Ł_h u)_i^n (q) + f_i^n (q) \right\} \Delta t - \rho u_i^n \Delta t \right) + \max \left\{ (\mathcal{M}_h u)_i^n - u_i^n, 0 \right\} / \epsilon$$

- Equation is nonlinear in $u^n$!
- Requires iterative method
- Instead of a penalization, we could use a direct control method. However, the proof of convergence (see [Azimzadeh and Forsyth, 2016]) is involved

# Numerical schemes

## Explicit control

## Explicit control

- After each timestep, controller applies either the stochastic or impulse control:

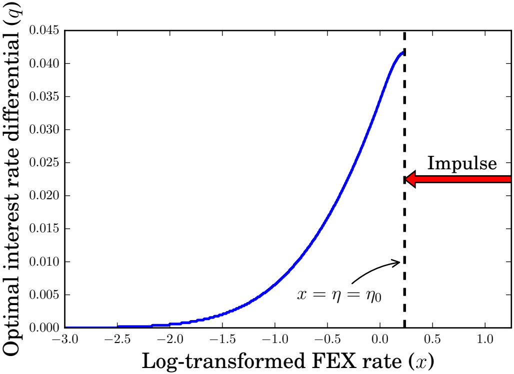
$$u_i^n = \left(\hat{\mathcal{L}}_h u\right)_i^n \Delta t - \rho u_i^n \Delta t$$
$$+ \max\left(\sup_{q \in Q_h}\left\{u^{n-1}\left(x_i + \mu_i^n\left(q\right)\Delta t\right) + f_i^{n-1}\left(q\right)\Delta t\right\}, \left(\mathcal{M}_h u\right)_i^{n-1}\right)$$

- Equation is linear in $u^n$!
- No iterative method needed

# Results

## Optimal control of the exchange rate

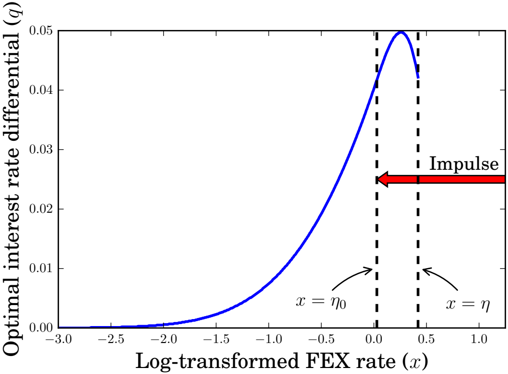
### Fixed cost $c = 0$



# Results

## Optimal control of the exchange rate

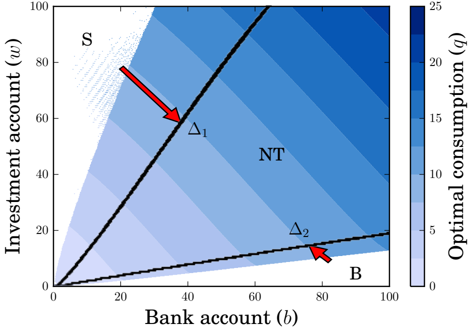
### Fixed cost $c > 0$



# Results

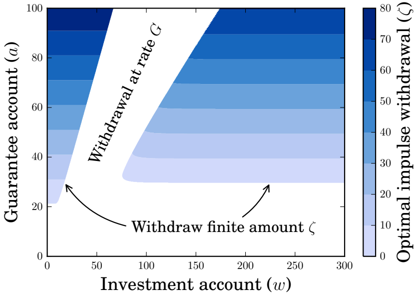
## Optimal consumption

## Optimal consumption ($c > 0$)



- Oksendal, Bernt, and Agnes Sulem. "Optimal consumption and portfolio with both fixed and proportional transaction costs." *SIAM Journal on Control and Optimization* 40.6 (2002): 1765-1790.

# Guaranteed minimum withdrawal benefits in variable annuities



- Dai, Min, Yue Kuen Kwok, and Jianping Zong. "Guaranteed minimum withdrawal benefit in variable annuities." *Mathematical Finance* 18.4 (2008): 595-611.

# QuantPDE

## Open-source C++ library for pricing derivatives

### QuantPDE

- Open-source C++ library for pricing derivatives
- `http://quantpde.org`
- HJBQVI module

# Optimal combined control of the exchange rate

```
HJBQVI<Dimension, StochasticControlDimension, ImpulseControlDimension> hjbqvi(
        { Axis::uniform( x_min, x_max, num_x_points ) }, // spatial grid
        { Axis::uniform( q_min, q_max, num_q_points ) }, // stochastic control grid
        { Axis::uniform( x_min, x_max, num_x_points ) }, // impulse control grid

        T, // expiry

        [=] (Real t, Real x) { return rho; }, // rho (discount)
        { [=] (Real t, Real x) { return sigma; } }, // sigma (volatility)

        // mu (drift)
        { [=] (Real t, Real x, Real q) { return -a * q; } },
        { [=] (Real t, Real x) { return 0.; } },

        // f (continuous penalty)
        [=] (Real t, Real x, Real q) { return -q * q * b; },
        [=] (Real t, Real x) { return -max(x, 0.) * max(x, 0.); },

        // Mu (impulse)
        { [=] (Real t, Real x, Real x_new) { return x_new; } },
        [=] (Real t, Real x, Real x_new) {
            const Real zeta = x_new - x
            return - lambda * fabs(zeta) - c;
        },

        [=] (Real t, Real x) { return 0.; }, // g (exit function)

        timesteps // Initial number of timesteps
);
```

# Thanks!

UNIVERSITY OF
WATERLOO

# Bibliography

[Azimzadeh and Forsyth, 2016] Azimzadeh, P. and Forsyth, P. A. (2016).
Weakly chained matrices, policy iteration, and impulse control.
*SIAM Journal on Numerical Analysis*, 54(3):1341–1364.