Convergence of approximation schemes for weakly nonlocal second order equations

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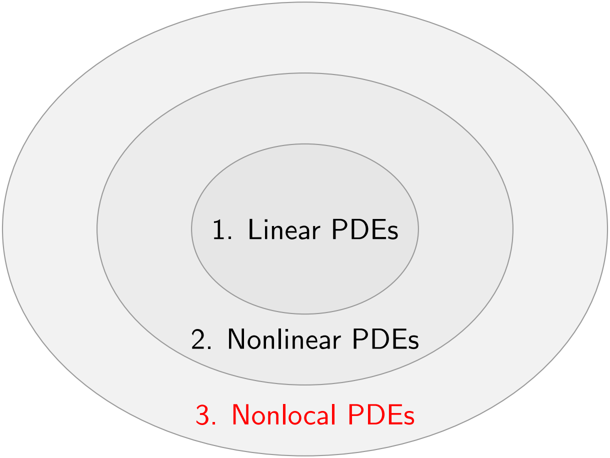
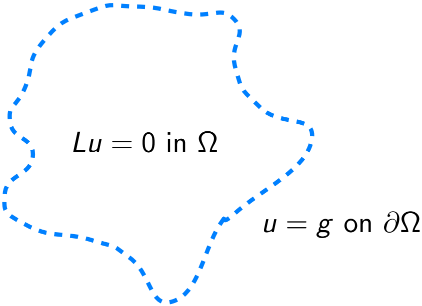
Outline



1. Linear PDEs

Boundary-value problem



Example: 1D heat equation

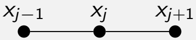
$$Lu \equiv -u_{xx}$$

Discretization

- To compute solution of BVP, discretize operator L

Example: second derivative stencil

$$u_{xx} \approx (\mathcal{D}_h^2 u)_j \equiv \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2} \text{ where } x_j = jh$$



Approximation scheme

Example: discretizing the 1D heat equation

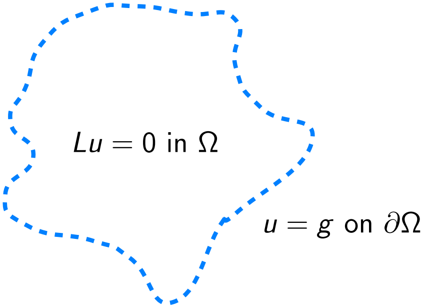
Discretization is summarized by a function S:

$$S(h, x_j, u) \equiv \begin{cases} -\dfrac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2} & \text{if } x_j \in \Omega \\ u(x_j) - g(x_j) & \text{if } x_j \in \partial\Omega \end{cases}$$

We call u_h a numerical solution if $S(h, x, u_h) = 0$ for all x

- We call S as an approximation scheme
- When is an approximation scheme convergent?

Lax equivalence theorem



Theorem: Lax equivalence

If S is a stable and consistent approximation scheme for a well-posed BVP, u_h converges to the solution u of the BVP

Lax-equivalence theorem (cont'd)

Definition: stability

S is stable if its solutions u_h are bounded independently of h

Definition: consistency

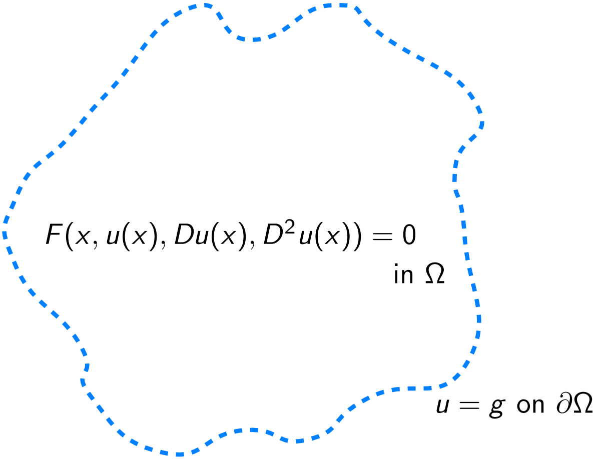
S is consistent if there is a $p \geq 1$ such that

$$S(h, x, \varphi) - L\varphi(x) = O(h^{1+p})$$

for each φ in an "appropriate" set of smooth functions [SSP85]

2. Nonlinear (2nd order) PDEs

Boundary-value problem



Examples

- Some examples are
  - optimal control (i.e., Hamilton–Jacobi–Bellman)
  - continuous-time games (i.e., Hamilton–Jacobi–Bellman–*Isaacs*)
  - motion by mean curvature
  - front propagation of surfaces
  - p-Laplacian
  - ...

Weak solution

- Nonlinear PDEs do not generally admit smooth solutions
- Require a relaxed, or “weak” notion of solution
- Viscosity solutions: right solution concept for many applications
  - (see examples on a previous slide)

Viscosity solution

- First, extend the def'n of F to incorporate boundary data g:

$$F(x, u(x), Du(x), D^2u(x)) \equiv u(x) - g(x) \text{ if } x \in \partial\Omega$$

Definition: viscosity solution (Barles–Souganidis)

A locally bdd function u is a viscosity subsolution (resp. supersolution) if for each smooth function φ such that $u - \varphi$ admits a maximum (resp. minimum) at x,

$$F_*(x, u^*(x), D\varphi(x), D^2\varphi(x)) \leq 0$$
$$(\text{resp. } F^*(x, u_*(x), D\varphi(x), D^2\varphi(x)) \geq 0)$$

u is a viscosity solution if it is a subsolution and supersolution

Barles–Souganidis framework [BS91]

Theorem: Barles–Souganidis framework

If S is a stable, consistent, and monotone approximation scheme for a nonlinear BVP satisfying a comparison principle, u_h converges to the solution u of the BVP

- Analogue of the Lax equivalence theorem for the nonlinear case
  - Comparison principle $\Rightarrow$ uniqueness

	Linear	Nonlinear
Stable	$\times$	$\times$
Consistent	$\times$	$\times$
Monotone		$\times$

Barles–Souganidis framework [BS91] (cont'd)

- Stability is the same as before:

Definition: stability

S is stable if its solutions u_h are bounded independently of h

Barles–Souganidis framework [BS91] (cont'd)

Definition: consistency

S is consistent if for all points x and smooth functions φ,

$$\liminf_{\substack{h\to 0\\ y\to x\\ \mathbb{R}\ni\xi\to 0}} S(h, y, \varphi + \xi) \leq F_*(x, \varphi(x), D\varphi(x), D^2\varphi(x))$$

$$\limsup_{\substack{h\to 0\\ y\to x\\ \mathbb{R}\ni\xi\to 0}} S(h, y, \varphi + \xi) \geq F^*(x, \varphi(x), D\varphi(x), D^2\varphi(x))$$

- If lim on LHS exists and F is continuous, inequalities become

$$\lim_{\substack{h\to 0\\ y\to x\\ \mathbb{R}\ni\xi\to 0}} S(h, y, \varphi + \xi) = F(x, \varphi(x), D\varphi(x), D^2\varphi(x))$$

Barles–Souganidis framework [BS91] (cont'd)

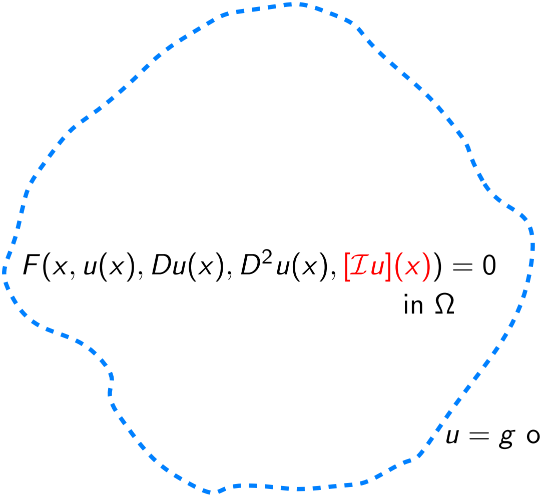
Definition: monotonicity

S is monotone if for all x and functions $u, \tilde{u}$ satisfying $\tilde{u} \geq 0$ and $\tilde{u}(x) = 0$,

$$S(\cdot, x, u) \geq S(\cdot, x, u + \tilde{u})$$

3. Nonlocal PDEs

Boundary-value problem

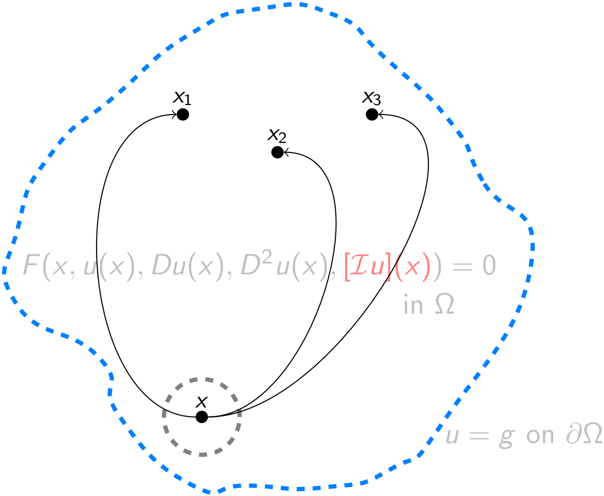


$$F(x, u(x), Du(x), D^2u(x), [\mathcal{I}u](x)) = 0$$

in Ω

$u = g$ on $\partial\Omega$

Boundary-value problem



Example

Example: intervention operator from impulse control

Intervention operator from impulse control:

$$[\mathcal{I}u](x) = \sup_{z \in Z(x)} \{u(x + \Gamma(x, z)) + K(x, z)\}$$

Viscosity solution

- First, define viscosity solution for nonlocal F
- Extend the def'n of F to incorporate boundary data g:

$$F(x, u(x), Du(x), D^2u(x), [\mathcal{I}u](x)) \equiv u(x) - g(x) \text{ if } x \in \partial\Omega$$

Viscosity solution (cont'd)

Definition: viscosity solution

A locally bdd function u is a viscosity subsolution if for each smooth function φ such that $u - \varphi$ admits a maximum at x,

$$F_*(x, u^*(x), D\varphi(x), D^2\varphi(x), [\mathcal{I}u^*](x)) \leq 0$$

(supersolutions are defined symmetrically).

Barles–Souganidis framework

- For certain nonlocal operators $\mathcal{I}$, we can still use the Barles–Souganidis framework
  - possible if a comparison principle holds under an alternate notion of viscosity solution in which $\mathcal{I}u$ is replaced by $\mathcal{I}\varphi$ on previous slide [CV05]
  - e.g., integro-differential operator [AT96]
- This is not always possible!
  - e.g., intervention operator

Modify def'n of approximation scheme

- We have to change our notion of approximation scheme
- Previously had $S \equiv S(h, x, u)$
- Now we have $S \equiv S(h, x, u, w)$

Main result

Theorem [ABL17]

If S is a stable, nonlocally consistent, and monotone approximation scheme for a nonlocal BVP satisfying a comparison principle, u_h converges to the solution u of the BVP

- Generalization of Barles–Souganidis result
- Consistency $\rightarrow$ nonlocal consistency

Main result (cont'd)

- Stability and monotonicity are the same as before:

Definition: stability

S is stable if its solutions u_h are bounded independently of h

Definition: monotonicity

S is monotone if for all x and functions $u, \tilde{u}$ satisfying $\tilde{u} \geq 0$ and $\tilde{u}(x) = 0$,

$$S(\cdot, x, u + \tilde{u}, \cdot) \leq S(\cdot, x, u, \cdot)$$

Main result (cont'd)

Definition: nonlocal consistency

S is nonlocally consistent if for each point x, smooth function φ, and family of uniformly bounded maps $(w_h)_{h>0}$ with half-relaxed limits $\overline{w} = \limsup_{\substack{h\to 0\\ y\to x}} w_h(y)$ and $\underline{w} = \liminf_{\substack{h\to 0\\ y\to x}} w_h(y)$,

$$\liminf_{\substack{h\to 0\\ y\to x\\ \mathbb{R}\ni\xi\to 0}} S(h, y, \varphi+\xi, w_h) \leq F_*(x, \varphi(x), D\varphi(x), D^2\varphi(x), [\mathcal{I}\overline{w}](x))$$

$$\limsup_{\substack{h\to 0\\ y\to x\\ \mathbb{R}\ni\xi\to 0}} S(h, y, \varphi+\xi, w_h) \geq F^*(x, \varphi(x), D\varphi(x), D^2\varphi(x), [\mathcal{I}\underline{w}](x))$$

Bottom line

Very general and simple result that handles nonlocality by using half-relaxed limits

Application

Example: intervention operator from impulse control

Intervention operator from impulse control:

$$[\mathcal{I}u](x) = \sup_{z \in Z(x)} \{u(x + \Gamma(x, z)) + K(x, z)\}$$

- In [ABL17], we study HJBQVIs involving intervention operators
- Create and prove convergence of a hybrid MC-PDE scheme that is
  - fully explicit,
  - unconditionally stable,
  - trivially monotone in higher dimensions, and
  - embarrassingly parallel

Hybrid MC-PDE scheme for HJBQVIs

$$\min\left\{u_t - {}^1\!/_2\,\mathrm{trace}(bb^\intercal D^2 u) - a \cdot Du - f,\, u - \mathcal{I}u\right\} = 0$$

↓

$$\min\left\{u_i^n - \mathbb{E}[u^{n-1}(X)] - f_i^n \Delta t,\, u_i^n - (\mathcal{I}u^{n-1})_i\right\} = 0$$

where

$$X = x_i + a(t^n, x_i)\Delta t + b(t^n, x_i) Z \sqrt{\Delta t}$$

and Z is a standard normal random variable.

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