All of Statistics - Chapter 8 Solutions

Jan 23, 2020

1.

TODO (Computer Experiment)

2.

TODO (Computer Experiment)

3.

TODO (Computer Experiment)

4.

This is a stars and bars problem (or, equivalently, an "indistinguishable balls in distinct buckets" problem). For example, the configuration $\star|\star\star\star||\star$ corresponds to sampling X_1 once, sampling X_2 three times, sampling X_3 zero times, and sampling X_4 once. In general, there are n stars and $n-1$ bars, and hence the total number of configurations is $(2n-1)!/(n!(n-1)!)$.

5.

First, note that

$$\mathbb{E}\left[\overline{X}_n^* \mid X_1, \ldots, X_n\right] = \mathbb{E}\left[X_1^* \mid X_1, \ldots, X_n\right] = \overline{X}_n.$$

Therefore, by the tower property, $\mathbb{E}[\overline{X}_n^*] = \mathbb{E}[X_1]$. Next, note that

$$\mathbb{V}(\overline{X}_n^* \mid X_1, \ldots, X_n) = \frac{1}{n}\mathbb{V}(X_1^* \mid X_1, \ldots, X_n) = \frac{1}{n^2}\sum_i \left(X_i - \overline{X}_n\right)^2.$$

The above can also be expressed as $S_n(n-1)/n^2$ where S_n is the unbiased sample variance of $(X_1, \ldots, X_n)$. Next, note that

$$\mathbb{E}\left[\left(\overline{X}_n\right)^2\right] = \frac{1}{n^2}\mathbb{E}\left[\sum_i X_i^2 + \sum_{i \neq j} X_i X_j\right] = \frac{1}{n}\left(\sigma^2 + \mu^2\right) + \frac{n-1}{n}\mu^2 = \frac{\sigma^2}{n} + \mu^2$$

where $\mu = \mathbb{E}[X_1]$ and $\sigma^2 = \mathbb{V}(X_1)$. Now, recall that for any random variable Y,

$$\mathbb{V}(Y \mid \mathcal{H}) = \mathbb{E}\left[Y^2 \mid \mathcal{H}\right] - \mathbb{E}[Y \mid \mathcal{H}]^2.$$

Therefore, by the tower property,

$$\mathbb{E}\left[Y^2\right] = \mathbb{E}\left[\mathbb{V}(Y \mid \mathcal{H}) + \mathbb{E}[Y \mid \mathcal{H}]^2\right].$$

Applying this to our setting,

$$\mathbb{E}\left[\left(\overline{X}_n^*\right)^2\right] = \mathbb{E}\left[\frac{n-1}{n^2}S_n + \left(\overline{X}_n\right)^2\right] = \frac{2n-1}{n^2}\sigma^2 + \mu^2.$$

As such, we can conclude that

$$\mathbb{V}(\overline{X}_n^*) = \frac{2n-1}{n^2}\sigma^2 = \frac{2n-1}{n}\mathbb{V}(\overline{X}_n) \sim 2\mathbb{V}(\overline{X}_n)$$

where the asymptotic is in the limit of large n.

6.

TODO (Computer Experiment)

7.

a)

The distribution of $\hat{\theta}$ is given in the solution of Question 2 of Chapter 6.

TODO (Computer Experiment)

b)

Let $\hat{\theta}^*$ be a bootstrap resample. Then,

$$\mathbb{P}(\hat{\theta}^* = \hat{\theta} \mid \hat{\theta}) = 1 - \mathbb{P}(\hat{\theta}^* \neq \hat{\theta} \mid \hat{\theta}) = 1 - (1 - 1/n)^n \to 1 - \exp(-1) \approx 0.632.$$

8.

TODO