All of Statistics - Chapter 6 Solutions

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1.

Since $\mathbb{E}_\lambda[\hat{\lambda}] = \mathbb{E}_\lambda[X_1]$, the estimator is unbiased. Moreover, $\mathrm{se}(\hat{\lambda})^2 = \mathbb{V}_\lambda(X_1)/n = \lambda/n$. By the bias-variance decomposition, the MSE is equal to $\mathrm{se}(\hat{\lambda})^2$.

2.

If y is between 0 and θ,

$$\mathbb{P}_\theta(\hat{\theta} \leq y) = \mathbb{P}_\theta(X_1 \leq y)^n = (y/\theta)^n.$$

Differentiating yields the PDF of $\hat{\theta}$ between 0 and θ as $y \mapsto n(y/\theta)^n/y$. Therefore,

$$\mathbb{E}_\theta[\hat{\theta}] = \int_0^\theta n(y/\theta)^n dy = \theta n/(n+1).$$

It follows that the bias of this estimator is $-\theta/(n+1)$ Moreover,

$$\mathrm{se}(\hat{\theta})^2 = \int_0^\theta ny(y/\theta)^n dy - \mathbb{E}_\theta[\hat{\theta}]^2 = \theta^2 n/(n+2) - \mathbb{E}_\theta[\hat{\theta}]^2.$$

By the bias-variance decomposition, the MSE is $\theta^2 n/(n+2) - \theta^2(n^2-1)/(n+1)^2$.

Remark. $\hat{\theta}(n+1)/n$ is an unbiased estimator.

3.

Since $\mathbb{E}_\theta[\hat{\theta}] = 2\mathbb{E}_\theta[X_1] = \theta$, the estimator is unbiased. Moreover,

$$\mathrm{se}(\hat{\theta})^2 = 4\mathbb{V}_\theta(X_1)/n = \theta^2/(3n).$$

By the bias-variance decomposition, the MSE is equal to $\mathrm{se}(\hat{\theta})^2$.