All of Statistics - Chapter 5 Solutions

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1.

a)

See Question 8 of Chapter 3.

b)

First, note that

$$S_n^2 = \frac{1}{n-1}\sum_i \left(X_i - \bar{X}\right)^2 = \frac{1}{n-1}\sum_i \left(X_i^2 - 2X_i\bar{X} + \bar{X}^2\right)$$
$$= \frac{1}{n-1}\sum_i X_i^2 - \frac{n}{n-1}\bar{X}^2 = c_n\frac{1}{n}\sum_i X_i^2 - d_n\bar{X}^2$$

where $c_n \to 1$ and $d_n \to 1$. By the WLLN, $n^{-1}\sum_i X_i^2$ and $\bar{X}^2$ converge, in probability, to $\mathrm{E}[X_1^2]$ and μ^2. By Theorem 5.5 (d), $c_n n^{-1}\sum_i X_i^2$ and $d_n\bar{X}^2$ converge, in probability, to the same quantities. Lastly, by Theorem 5.5 (a), S_n^2 converges, in probability, to $\mathrm{E}[X_1^2] - \mu^2 = \sigma^2$.

2.

Suppose X_n converges to b in quadratic mean. By Jensen's inequality,

$$\mathrm{E}\left[X_n - b\right]^2 \le \mathrm{E}\left[\left|X_n - b\right|\right]^2 \le \mathrm{E}[(X_n - b)^2] \to 0$$

Therefore, $\mathrm{E}X_n \to b$. Next, note that

$$\mathrm{E}[(X_n - b)^2] = \mathrm{E}\left[X_n^2\right] - 2b\mathrm{E}\left[X_n\right] + b^2 = \mathrm{V}(X_n) + \mathrm{E}[X_n]^2 - 2b\mathrm{E}\left[X_n\right] + b^2$$

Taking limits of both sides reveals $\lim_n \mathrm{V}(X_n) = 0$. As for the converse, we can apply the limits $\lim_n \mathrm{E}[X_n] = b$ and $\lim_n \mathrm{V}(X_n) = 0$ directly to the equation above.

3.

Since the expectation of $\bar{X}$ is μ and the variance of $\bar{X}$ converges to zero, the desired result is obtained by an application of our findings in Problem 2.

Alternatively, taking a more direct approach, note that

$$\mathrm{E}\left[\left(\bar{X}-\mu\right)^2\right] = \mathrm{E}\left[\bar{X}^2 - 2\mu\bar{X} + \mu^2\right] = \mathrm{E}\left[\bar{X}^2\right] - \mu^2$$

$$= \frac{1}{n^2}\mathrm{E}\left[\sum_i X_i^2 + \sum_{i,j:i\neq j} X_i X_j\right] - \mu^2 = \frac{1}{n}\mathrm{E}\left[X_1^2\right] + \frac{n-1}{n}\mathrm{E}\left[X_1 X_2\right] - \mu^2.$$

Taking the limit,

$$\mathrm{E}[(\bar{X}-\mu)^2] \to \mathrm{E}\left[X_1 X_2\right] - \mu^2 = \mathrm{E}\left[X_1\right]\mathrm{E}\left[X_2\right] - \mu^2 = 0.$$

4.

Let $\epsilon > 0$. For n sufficiently large,

$$\mathrm{P}(\left|X_n - 0\right| > \epsilon) = \mathrm{P}(X_n > \epsilon) = \mathrm{P}(X_n = n) = 1/n^2 \to 0$$

and hence X_n converges in probability. However,

$$\mathrm{E}\left[\left(X_n - 0\right)^2\right] = \mathrm{E}\left[X_n^2\right] \geq \mathrm{E}\left[X_n^2 I_{\{X_n = n\}}\right] = n^2\mathrm{P}(X_n = n) = 1$$

and hence X_n does not converge in quadratic mean.

5.

It is sufficient to prove the second claim since convergence in quadratic mean implies convergence in probability. Similarly to Problem 3, we can define $Y_i = X_i^2$ and apply our findings in Problem 2 to $\bar{Y}$.

Alternatively, taking a more direct approach, note that

$$\left(\frac{1}{n}\sum_i X_i^2 - p\right)^2 = \frac{1}{n^2}\sum_i X_i^4 + \frac{1}{n^2}\sum_{i,j:i\neq j} X_i^2 X_j^2 - \frac{2}{n}p\sum_i X_i^2 + p^2.$$

Taking expectations, and using the fact that $X_i^k = X_i$ and $\mathrm{E}X_i = p$,

$$\mathrm{E}\left[\left(\frac{1}{n}\sum_i X_i^2 - p\right)^2\right] = \frac{1}{n}\mathrm{E}\left[X_1^4\right] + \frac{n-1}{n}\mathrm{E}\left[X_1^2\right]\mathrm{E}\left[X_2^2\right] - 2p\mathrm{E}\left[X_1^2\right] + p^2$$

$$= \frac{1}{n}p + \frac{n-1}{n}p^2 - p^2 \to p^2 - p^2 = 0.$$

6.

Letting F denote the CDF of a standard normal distribution, by the CLT,

$$\mathrm{P}\left(\frac{X_1 + \cdots + X_{100}}{100} \geq 68\right)$$

$$= \mathrm{P}\left(\frac{\sqrt{100}}{2.6}\left(\frac{X_1 + \cdots + X_{100}}{100} - 68\right) \geq 0\right) \approx 1 - F(0) = 0.5.$$

7.

Let $f > 0$ be a function and $\epsilon > 0$ be a constant. Then,

$$\mathrm{P}(\left|f(n)X_n - 0\right| > \epsilon) = \mathrm{P}(X_n > \epsilon / f(n)) \leq \mathrm{P}(X_n \neq 0) = 1 - \exp(-1/n) \to 0.$$

It follows that $f(n)X_n$ converges to zero in probability. Take $f = 1$ for Part (a) and $f(n) = n$ for (b).

8.

Letting F denote the CDF of a standard normal distribution, by the CLT,

$$\mathrm{P}(Y < 90) = \mathrm{P}(X_1 + \cdots + X_{100} < 90)$$

$$= \mathrm{P}\left(\frac{\sqrt{100}}{1}\left(\frac{X_1 + \cdots + X_{100}}{100} - 1\right) < -1\right) \approx F(-1)$$

9.

Let $\epsilon > 0$. Then,

$$\mathrm{P}(\left|X_n - X\right| > \epsilon) \leq \mathrm{P}(X_n \neq X) = 1/n \to 0.$$

Therefore, X_n converges in probability (and hence in distribution) to X. On the other hand,

$$\mathrm{E}\left[\left(X - X_n\right)^2\right] = \mathrm{E}\left[\left(X - e^n\right)^2 I_{\{X_n \neq X\}}\right]$$

$$= \mathrm{E}\left[1 - 2Xe^n + e^{2n}\right]\mathrm{P}(X_n \neq X) = \frac{1 + e^{2n}}{n} \to \infty.$$

10.

Since $1 \le x^k/t^k$ whenever $x \ge t > 0$, it follows that

$$P(Z > t) = E\left[I_{\{Z>t\}}\right] \le E\left[I_{\{Z>t\}}\left(\frac{Z}{t}\right)^k\right] \le \frac{E\left[I_{\{Z>t\}}|Z|^k\right]}{t^k}$$

Therefore, since the distribution is symmetric,

$$P(|Z| > t) = 2P(Z > t) \le \frac{E\left[|Z|^k\left(I_{\{Z>t\}} + I_{\{Z<-t\}}\right)\right]}{t^k} \le \frac{E|Z|^k}{t^k}.$$

Note that we only used symmetry in establishing the above and hence the result is more general than the problem description implies.

11.

First, note that X is almost surely zero. Let $\epsilon > 0$ and Z be a standard normal random variable. Then,

$$P(\left|X_n - X\right| > \epsilon) = P(\left|X_n\right| > \epsilon) = P(|Z| > \epsilon\sqrt{n}) \le \frac{E\left[Z^2\right]}{\epsilon^2 n} = \frac{1}{\epsilon^2 n} \to 0.$$

Therefore, X_n converges in probability (and hence in distribution) to zero.

12.

Let F be the CDF of an integer valued random variable K. Let k be an integer. It follows that $F(k) = F(k + c)$ for all $0 \le c < 1$. We use this observation multiple times below.

To prove the forward direction, suppose $X_n \rightsquigarrow X$. By definition, $F_{X_n} \to F_X$ at all points of continuity of F_X. Therefore,

$$\begin{aligned} P(X_n = k) = F_{X_n}(k + 1/2) - F_{X_n}(k - 1/2) &\to F_X(k + 1/2) - F_X(k - 1/2) \\ &= P(X = k). \end{aligned}$$

To prove the reverse direction, suppose $P(X_n = k) \to P(X = k)$ for all integers k. Let j be an integer and note that

$$F_{X_n}(j) = \sum_{k \le j} P(X_n = k) \to \sum_{k \le j} P(X = k) = F_X(j)$$

and hence $X_n \rightsquigarrow X$ as desired.

13.

First, note that

$$F_{X_n}(x) = \mathrm{P}(\min\left\{Z_1, \ldots, Z_n\right\} \leq x/n) = 1 - \mathrm{P}(Z_1 \geq x/n)^n.$$

If $x \leq 0$, it follows that $F_{X_n}(x) = 0$. Otherwise,

$$\mathrm{P}(Z_1 \geq x/n)^n = \left(1 - \mathrm{P}(Z_1 \leq x/n)\right)^n = \left(1 - \int_0^{x/n} f(z)dz\right)^n$$

$$= \left(1 - f(c_n)\frac{x}{n}\right)^n = \left(e^{-f(c_n)x/n} + O(n^{-2})\right)^n \to e^{-\lambda x}.$$

Therefore, $F_{X_n}(x) \to (1 - e^{-\lambda x})I_{(0,\infty)}(x)$ and hence X_n converges in distribution to an Exp(λ) random variable.

14.

By the CLT

$$\frac{\sqrt{n}}{\sigma}\left(\bar{X} - \mu\right) = \frac{\sqrt{n}}{1/\sqrt{12}}\left(\bar{X} - \frac{1}{2}\right) \rightsquigarrow N(0, 1).$$

Let $g(x) = x^2$ so that $g'(x) = 2x$. By the delta method,

$$\frac{\sqrt{n}}{\left|g'(\mu)\right|\sigma}\left(g(\bar{X}) - g(\mu)\right) = \frac{\sqrt{n}}{1/\sqrt{12}}\left(Y_n - \frac{1}{4}\right) \rightsquigarrow N(0, 1).$$

15.

Define $g : \mathrm{R}^2 \to \mathrm{R}$ by $g(x) = x_1/x_2$. Then, $\nabla g(x) = (1/x_2, \ -x_1/x_2^2)^\intercal$. Define $\nabla_\mu = \nabla g(\mu)$ for brevity. By the multivariate delta method,

$$\sqrt{n}\left(Y_n - \frac{\mu_1}{\mu_2}\right) \rightsquigarrow N(0, \nabla_\mu^\intercal \Sigma \nabla_\mu) = N(0, \Sigma_{11}/\mu_2^2 - 2\Sigma_{12}\mu_1/\mu_2^3 + \Sigma_{22}\mu_1^2/\mu_2^4).$$

16.

Let $X_n, X, Y \sim N(0, 1)$ be IID with $X_n = Y_n$. Trivially, $X_n \rightsquigarrow X$ and $Y_n \rightsquigarrow Y$. However, $\mathrm{V}(X_n + Y_n) = \mathrm{V}(2X_n) = 4$ while $\mathrm{V}(X + Y) = 2$ and hence $X_n + Y_n$ does not converge in distribution to $X + Y$.