Math 525: Assignment 7 Solutions

1. We know X_n, Y_n converge in distribution to $X, Y \sim \text{Poisson}(\lambda)$. Therefore,

$$\phi_{Z_n}(t) = \mathbb{E}\left[e^{it(X_n - Y_n)}\right] = \mathbb{E}\left[e^{itX_n}\right]\mathbb{E}\left[e^{-itY_n}\right] = \phi_{X_n}(t)\phi_{Y_n}(-t) \to \phi_X(t)\phi_Y(-t)$$

by one direction of Lévy's continuity theorem. Therefore,

$$\phi_X(t)\phi_Y(-t) = \exp(\lambda(e^{it}-1))\exp(\lambda(e^{-it}-1)) = \exp\left(\lambda\left(e^{it} - e^{-it}\right)\right) = \exp\left(2\lambda\left(\cosh t - 1\right)\right).$$

By the other direction of Lévy's continuity theorem, Z_n converges in distribution to some random variable Z with the above characteristic function $\phi_X\phi_Y$, as desired.

2. See Exercise 2.2 of Lecture 15.

3. To show that $((X_n, X_{n+1}))_{n\geq 0}$ is a Markov chain, note that

$$\begin{aligned}
&\mathbb{P}\left((X_n, X_{n+1}) = (i_n, i_{n+1}) \mid (X_0, X_1) = (i_0, i_1), \ldots, (X_{n-1}, X_n) = (i_{n-1}, i_n)\right) \\
&\quad = \mathbb{P}\left(X_{n+1} = i_{n+1} \mid X_0 = i_0, \ldots, X_n = i_n\right) = \mathbb{P}\left(X_{n+1} = i_{n+1} \mid X_n = i_n\right) \\
&\quad = \mathbb{P}\left(X_{n+1} = i_{n+1} \mid X_{n-1} = i_{n-1}, X_n = i_n\right) \\
&\quad = \mathbb{P}\left((X_n, X_{n+1}) = (i_n, i_{n+1}) \mid (X_{n-1}, X_n) = (i_{n-1}, i_n)\right).
\end{aligned}$$

This follows from

4.

   (a) If P is a transition matrix, then

$$\sum_j (P^2)_{ij} = \sum_j \sum_k P_{ik}P_{kj} = \sum_k \sum_j P_{ik}P_{kj} = \sum_k P_{ik} \sum_j P_{kj} = \sum_k P_{ik} = 1.$$

   (b) If P is a bistochastic matrix, then P and $P^\intercal$ are transition matrices. Then, by part (a), P^2 and $(P^\intercal)^2 = (P^2)^\intercal$ are transition matrices. Therefore, P^2 is bistochastic.

5.

   (a) The multiplicity of the eigenvalue is 2.

   (b) The multiplicity of the eigenvalue is 1.

   (c) An "end" state of the game is called an *absorbing state.* Since there is a walk from an any state u to an absorbing state v, the multiplicity of the eigenvalue 1 tells us how many absorbing states there are.