Math 525: Assignment 5 Solutions

1. This is just a consequence of Cauchy-Schwarz:

$$\mathbb{E}\left[XYZ\right] \leq \sqrt{\mathbb{E}\left[(XY)^2\right]\mathbb{E}\left[Z^2\right]} = \sqrt{\mathbb{E}\left[X^2\right]\mathbb{E}\left[Y^2\right]\mathbb{E}\left[Z^2\right]}.$$

   We have used independence in the last equality.

2. Since $x \mapsto e^{\theta x}$ is convex, this is just a consequence of Jensen's inequality:

$$e^{\theta \mathbb{E}X} \leq \mathbb{E}\left[e^{\theta X}\right] = M(\theta).$$

3. Let $\epsilon > 0$ and

$$\Lambda_n^\epsilon = \left\{|X_n|^p \geq \epsilon\right\}.$$

   By Chebyshev's inequality,

$$\mathbb{P}(\Lambda_n) \leq \frac{1}{\epsilon^p}\mathbb{E}\left[|X_n|^p\right] \leq \frac{1}{\epsilon^p} f(n).$$

   Therefore, $\sum_n \mathbb{P}(\Lambda_n) < \infty$ and hence by Borel-Cantelli,

$$\mathbb{P}(\limsup_n \Lambda_n^\epsilon) = 0.$$

   Now, consider

$$\Lambda = \bigcup_{\substack{\epsilon > 0 \\ \epsilon \in \mathbb{Q}}} \left(\limsup_n \Lambda_n^\epsilon\right).$$

   If $\omega \notin \Lambda$, $X_n(\omega) \to 0$, as desired.