Math 525: Assignment 1 Solutions

1. To simplify notation, let

$$\mathcal{M}(\mathcal{G}) = \{\mathcal{F} \colon \mathcal{F} \text{ is a } \sigma - \text{algebra on } \Omega \text{ and } \mathcal{G} \subset \mathcal{F}\}$$

so that

$$\sigma(\mathcal{G}) = \bigcap_{\mathcal{F} \in \mathcal{M}(\mathcal{G})} \mathcal{F}.$$

Note that as a trivial consequence of the definition, $\sigma(\mathcal{G}) \supset \mathcal{G}$.

   (a) We claim that the intersection $\mathcal{F} = \cap_\alpha \mathcal{F}_\alpha$ of σ-algebras $\mathcal{F}_\alpha$ is itself a σ-algebra. Since $\sigma(\mathcal{G})$ is, by definition, the intersection of σ-algebras, the desired result follows from this claim. The claim is established by three points: *(i)* $\emptyset \in \mathcal{F}$ because $\emptyset \in \mathcal{F}_\alpha$ for each α. *(ii)* Suppose $A \in \mathcal{F}$. Then, $A \in \mathcal{F}_\alpha$ for each α and hence $A^c \in \mathcal{F}_\alpha$ for each α, from which it follows that $A^c \in \mathcal{F}$. *(iii)* Suppose $A_1, A_2, \ldots \in \mathcal{F}$. Then, $A_1, A_2, \ldots \in \mathcal{F}_\alpha$ for each α and hence $\cup_{n \geq 1} A_n \in \mathcal{F}_\alpha$ for each α, from which it follows that $\cup_{n \geq 1} A_n \in \mathcal{F}$.

   (b) Suppose $\mathcal{G} \subset \mathcal{G}'$. Since any σ-algebra which contains $\mathcal{G}'$ must also contain $\mathcal{G}$, we have $\mathcal{M}(\mathcal{G}) \supset \mathcal{M}(\mathcal{G}')$, from which the desired result follows.

   (c) Suppose $\mathcal{F}$ is a σ-algebra. Since $\mathcal{F} \in \mathcal{M}(\mathcal{F})$, it follows that $\sigma(\mathcal{F}) \subset \mathcal{F}$. Since we already know $\sigma(\mathcal{F}) \supset \mathcal{F}$, the desired result follows.

   (d) If $\mathcal{G} \subset \mathcal{F}$, part (b) tells us $\sigma(\mathcal{G}) \subset \sigma(\mathcal{F})$. If $\mathcal{F}$ is a σ-algebra, part (c) tells us $\sigma(\mathcal{F}) = \mathcal{F}$. Combining these two points, the desired result follows.

2. Checking that these are algebras is straightforward, so I will just point out why they are not σ-algebras:

   (a) Consider $\mathbb{N} = \{1, 2, \ldots\}$. Neither $\mathbb{N}$ nor $\mathbb{N}^c = \mathbb{R} \setminus \mathbb{N}$ are finite subsets of $\mathbb{R}$, and hence $\mathbb{N}$ is not in the algebra. However, we can write $\mathbb{N}$ as a countable union of elements of the algebra: $\mathbb{N} = \cup_{n \geq 1} \{n\}$.

   (b) The set $(-\infty, 0)$ can be written as a countable union of elements of the algebra: $(-\infty, 0) = \cup_{n \geq 1} (-\infty, -1/n]$.

3. Suppose the claim holds for n. We attempt to establish it for $n + 1$. Note that

$$\begin{aligned}\mathbb{P}(A_1 \cup \cdots \cup A_{n+1}) &= \mathbb{P}((A_1 \cup \cdots \cup A_n) \cup A_{n+1}) \\ &= \mathbb{P}(A_1 \cup \cdots \cup A_n) + \mathbb{P}(A_{n+1}) - \mathbb{P}((A_1 \cup \cdots \cup A_n) \cap A_{n+1})\end{aligned}$$

Let's handle each term in the sum separately. First, note that by our induction hypothesis,

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) \\ = \sum_{i \le n} \mathbb{P}(A_i) - \sum_{i<j\le n} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k\le n} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1}\mathbb{P}(A_1 \cap \cdots \cap A_n).$$

Similarly, applying our induction hypothesis,

$$\begin{aligned} \mathbb{P}((A_1 \cup \cdots \cup A_n) \cap A_{n+1}) &= \mathbb{P}((A_1 \cap A_{n+1}) \cup \cdots \cup (A_n \cap A_{n+1})) \\ &= \sum_{i \le n} \mathbb{P}(A_i \cap A_{n+1}) - \sum_{i<j\le n} \mathbb{P}(A_i \cap A_j \cap A_{n+1}) + \sum_{i<j<k\le n} \mathbb{P}(A_i \cap A_j \cap A_k \cap A_{n+1}) \\ &\qquad - \cdots + (-1)^{n+1}\mathbb{P}(A_1 \cap \cdots \cap A_n \cap A_{n+1}). \end{aligned}$$

Now, we can simplify our expression for $\mathbb{P}(A_1 \cup \cdots \cup A_{n+1})$:

$$\begin{aligned} &\mathbb{P}(A_1 \cup \cdots \cup A_{n+1}) \\ &\qquad = \sum_{i \le n+1} \mathbb{P}(A_i) - \sum_{i<j\le n+1} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k\le n+1} \mathbb{P}(A_i \cap A_j \cap A_k) \\ &\qquad\qquad - \cdots + (-1)^{n+2}\mathbb{P}(A_1 \cap \cdots \cap A_{n+1}). \end{aligned}$$

To finish the proof, note that the claim holds trivially for $n = 1$ (base case).