Math 525: Lecture 8

January 30, 2018

1 Moment inequalities

There are a few useful inequalities concerning moments of random variables we should cover. We start with Markov's inequality.

1.1 Markov's inequality

The following inequality is a special case of a more general measure theoretic result. In the measure theoretic setting, it is called Chebyshev's inequality.

Proposition 1.1 (Markov's inequality). Let p > 0, $\lambda > 0$, and X be a random variable with X^p integrable. Then,

$$\mathbb{P}(\{|X| \ge \lambda\}) \le \frac{1}{\lambda^p} \mathbb{E}\left[|X|^p\right].$$

Proof. First, note that

$$\mathbb{P}(\{|X| \ge \lambda\}) = \mathbb{P}(\{|X|^p \ge \lambda^p\}) = \mathbb{E}\left[I_{\{|X|^p \ge \lambda^p\}}\right]$$

But if $|X(\omega)|^p \ge \lambda^p$, then $1 \le |X(\omega)|^p / \lambda^p$. Therefore,

$$\mathbb{E}\left[I_{\{|X|^{p} \ge \lambda^{p}\}}\right] \le \mathbb{E}\left[\frac{|X|^{p}}{\lambda^{p}}I_{\{|X|^{p} \ge \lambda^{p}\}}\right] \le \frac{1}{\lambda^{p}}\mathbb{E}\left[|X|^{p}I_{\{|X|^{p} \ge \lambda^{p}\}}\right] \le \frac{1}{\lambda^{p}}\mathbb{E}\left[|X|^{p}\right].$$

Corollary 1.2. Let $\lambda > 0$ and Y be a square integrable (i.e., Y^2 is integrable) random variable. Then,

$$\mathbb{P}(\{|Y - \mathbb{E}Y| \ge \lambda\}) \le \frac{1}{\lambda^2} \operatorname{Var}(Y).$$

Proof. Take p = 2 and $X = Y - \mathbb{E}Y$ in Markov's inequality.

1.2 Cauchy-Schwarz(-Buniakovski) inequality

Proposition 1.3 (Cauchy-Schwarz(-Buniakovski) inequality). Let X and Y be square integrable random variables. Then,

$$\mathbb{E}\left[XY\right] \le \sqrt{\mathbb{E}\left[X^2\right]\mathbb{E}\left[Y^2\right]}.$$

Proof. If either X or Y is zero a.s., then the inequality is trivial. Therefore, suppose that neither is zero a.s. Let $\lambda \ge 0$. Then,

$$0 \le \mathbb{E}\left[(X - \lambda Y)^2 \right] = \mathbb{E}\left[X^2 \right] - 2\lambda \mathbb{E}\left[XY \right] + \lambda^2 \mathbb{E}\left[Y^2 \right]$$

and hence

$$\mathbb{E}\left[XY\right] \leq \frac{1}{2} \left(\frac{1}{\lambda} \mathbb{E}\left[X^2\right] + \lambda \mathbb{E}\left[Y^2\right]\right).$$

Letting $\lambda = \sqrt{\mathbb{E}[X^2]} / \sqrt{\mathbb{E}[Y^2]}$ yieldsw

$$\mathbb{E}\left[XY\right] \leq \frac{1}{2} \left(\frac{\sqrt{\mathbb{E}\left[Y^2\right]}}{\sqrt{\mathbb{E}\left[X^2\right]}} \mathbb{E}\left[X^2\right] + \frac{\sqrt{\mathbb{E}\left[X^2\right]}}{\sqrt{\mathbb{E}\left[Y^2\right]}} \mathbb{E}\left[Y^2\right]\right)$$
$$= \frac{\mathbb{E}\left[Y^2\right] \mathbb{E}\left[X^2\right]}{\sqrt{\mathbb{E}\left[X^2\right] \mathbb{E}\left[Y^2\right]}}$$
$$= \sqrt{\mathbb{E}\left[X^2\right] \mathbb{E}\left[Y^2\right]}.$$

Example 1.4. Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Bernoulli}(p)$ be random variables. Then, by the Cauchy-Schwarz inequality,

$$\mathbb{E}[XY] \le \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]} = \sqrt{\lambda(\lambda+1)p}.$$

If the two random variables are independent, then

 $\mathbb{E}\left[XY\right] = \mathbb{E}X\mathbb{E}Y = \lambda p.$

Indeed, you can check that for all $0 \le p \le 1$ and $\lambda \ge 0$,

$$\lambda p \leq \sqrt{\lambda (\lambda + 1) p}.$$

2 Jensen's inequality

Next, we will cover Jensen's inequality, probably one of the most useful inequalities in probability theory! To discuss Jensen's inequality, we need to recall the notion of a convex function:

Definition 2.1. Let X be a subset of \mathbb{R}^n . We say X is *convex* if for all points $x, y \in X$ and $\theta \in [0, 1]$, we have $\theta x + (1 - \theta)y \in X$.

Definition 2.2. Let X be convex and $f: X \to \mathbb{R}$. We call the set

$$epi(f) = \{(x, \mu) \in X \times \mathbb{R} \colon f(x) \le \mu\}$$

the epigraph of f. We say f is a convex function if its epigraph is convex. We say f is concave if -f is convex.

Intuitively, a convex function is one whose epigraph makes a "bowl":

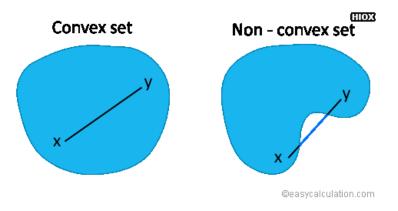


Figure 1: Examples of convex and non-convex sets

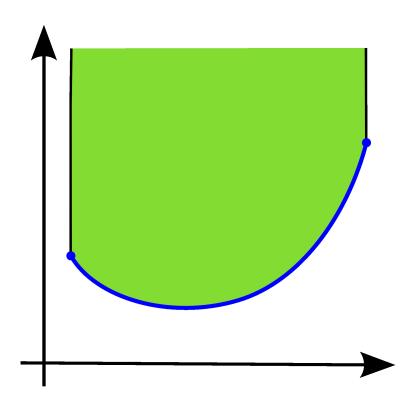


Figure 2: Epigraph (green) of a function f (blue curve)

Example 2.3. $x, |x|, x^2, e^{-x}$ are convex on \mathbb{R} . The function f defined by

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0\\ 0 & \text{if } x = 0 \end{cases}$$

is not convex on \mathbb{R} , but it is convex on $(-\infty, 0)$ and $(0, \infty)$.

Proposition 2.4. Let X be convex and $f: X \to \mathbb{R}$. f is a convex function if and only if for all $x, y \in X$ and $\theta \in [0, 1]$,

$$f(\theta x + (1 - \theta) y) \le \theta f(x) + (1 - \theta) f(y).$$

Proof. Suppose f is a convex function. Let $x, y \in X$ and $\theta \in [0, 1]$. Note that (x, f(x)) and (y, f(y)) are both points in epi(f). By convexity,

$$\theta(x, f(x)) + (1 - \theta)(y, f(y)) = (\theta x + (1 - \theta)y, \theta f(x) + (1 - \theta)f(y)) \in epi(f)$$

and hence

$$f(\theta x + (1 - \theta) y) \le \theta f(x) + (1 - \theta) f(y),$$

as desired.

Suppose f satisfies the convexity inequality. Let (x, μ_x) and (y, μ_y) be points in epi(f) and $\theta \in [0, 1]$. By the convexity inequality,

$$f(\theta x + (1 - \theta) y) \le \theta f(x) + (1 - \theta) f(y) \le \theta \mu_x + (1 - \theta) \mu_y$$

and hence

$$\theta(x,\mu_x) + (1-\theta)(y,\mu_y) = (\theta x + (1-\theta)y, \theta \mu_x + (1-\theta)\mu_y) \in \operatorname{epi}(f),$$

as desired.

Proposition 2.5. A convex set in \mathbb{R} is an interval.

Proof. Suppose $X \subset \mathbb{R}$ is convex and not an interval. Let $x, z \in X$ and y be such that x < y < z. Pick

$$\theta = \frac{z - y}{z - x}.$$

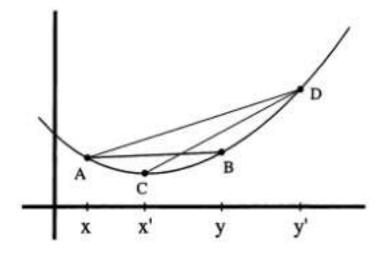
Then,

$$\theta x + (1 - \theta) y = \frac{z - y}{z - x} x + \frac{y - x}{z - x} z = \frac{xz - xy}{z - x} + \frac{yz - xz}{z - x} = \frac{yz - xy}{z - x} = y.$$

Proposition 2.6. Let I be an interval and $f: I \to \mathbb{R}$. Then, f is convex if and only if for all points $x, y, x', y' \in I$ such that $x \leq x' < y'$ and $x < y \leq y'$,

$$\frac{f(y) - f(x)}{y - x} \le \frac{f(y') - f(x')}{y' - x'}.$$

Proof. Suppose f is convex and let A = (x, f(x)), B = (y, f(y)), C = (x', f(x')), and D = (y', f(y')). Then... (proof by picture)



For the converse, let $x_1, x_2 \in I$ and $\theta \in [0, 1]$. Take $x = x_1, y' = x_2$, and $y = x' = \theta x_1 + (1 - \theta) x_2$ to get

$$\frac{f(\theta x_1 + (1 - \theta)x_2) - f(x_1)}{\theta x_1 + (1 - \theta)x_2 - x_1} \le \frac{f(x_2) - f(\theta x_1 + (1 - \theta)x_2)}{x_2 - \theta x_1 - (1 - \theta)x_2}$$

and hence

$$\theta \left(f(\theta x_1 + (1 - \theta) x_2) - f(x_1) \right) (x_2 - x_1) \le (1 - \theta) \left(f(x_2) - f(\theta x_1 + (1 - \theta) x_2) \right) (x_2 - x_1).$$

Simplifying,

$$f(\theta x_1 + (1 - \theta)x_2) \le \theta f(x_1) + (1 - \theta) f(x_2).$$

Corollary 2.7. Let I = (a, b) be an open interval and $f: I \to \mathbb{R}$ be a convex function. Then, f is continuous and the left and right derivatives

$$D_{-}f(x) = \lim_{h \downarrow 0} \frac{f(x) - f(x-h)}{h} \quad and \quad D_{+}f(x) = \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}$$

exist at each point $x \in I$. Moreover, D_{-f} and D_{+f} are nondecreasing with $D_{-f} \leq D_{+f}$.

Proof. Let x be a point in I. By Proposition 2.6, for all h > 0 such that x - h and x + h are points in I,

$$\frac{f(x) - f(x-h)}{h} \le \frac{f(x+h) - f(x)}{h}.$$
 (1)

We would like to take limits and conclude

$$D_{-}f(x) = \lim_{h \downarrow 0} \frac{f(x) - f(x-h)}{h} \le \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{h} = D_{+}f(x)$$

But first, we have to show these limits exist: note that Proposition 2.6 implies that the left hand side of (1) increases while the right hand side of (1) decreases as h is made smaller. That is, for a decreasing sequence of $(h_n)_n$ with $h_n \downarrow 0$,

$$\frac{f(x) - f(x - h_1)}{h} \le \frac{f(x) - f(x - h_2)}{h} \le \dots \le \frac{f(x + h_2) - f(x)}{h} \le \frac{f(x + h_1) - f(x)}{h}$$

Then, the limits exist by the monotone convergence theorem (recall that the monotone convergence theorem for sequences says that if a sequence is nondecreasing and bounded above, it must have a limit). \Box

Proposition 2.8. Let I = (a, b) be an open interval and $f: I \to \mathbb{R}$ be a convex function. Then, for each $x_0 \in I$, there exists m such that for all $x \in I$,

$$f(x) \ge f(x_0) + m(x - x_0).$$

That is, at each point x_0 , there exists a supporting line.

This fact generalizes to higher dimensions, in which case the supporting line becomes a **supporting hyperplane**.

Proof. Choose m such that

$$D_-f(x_0) \le m \le D_+f(x_0).$$

Now, if $x > x_0$,

$$m \le D_+ f(x_0) = \lim_{h \downarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \le \frac{f(x) - f(x_0)}{x - x_0}$$

and hence

$$f(x_0) + m(x - x_0) \le f(x).$$

The case of $x < x_0$ is identical (use $D_-f(x_0)$ in the argument).

Proposition 2.9 (Jensen's inequality). Let I = (a, b) be an open interval and $f: I \to \mathbb{R}$ be a convex function. Let X be a random variable which takes values in (a, b) a.s. If X and $f \circ X$ are both integrable,

$$\mathbb{E}\left[f(X)\right] \ge f(\mathbb{E}X).$$

Proof. Let $x_0 = \mathbb{E}X$. Now, we can find some supporting line parameterized by m:

$$f(x) \ge f(x_0) + m(x - x_0).$$

Substitute x = X to get

$$f(X) \ge f(x_0) + m(X - x_0)$$

(this inequality holds only a.s.). Take expectations of both sides to get

$$\mathbb{E}\left[f(X)\right] \ge \mathbb{E}\left[f(x_0) + m(X - x_0)\right] = \mathbb{E}\left[f(x_0)\right] = f(x_0) = f(\mathbb{E}X).$$